

Spin and the Symplectic Flag Manifold

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Abstract

A theory for the transitive action of a group on the configuration space of a system of particles is shown to lead to the conclusion that interactions can be represented by the action of cosets of the group. By application of this principle to Pauli spinors, the symplectic group $Sp(n)$ is shown to be the largest group of isometries of the space. Interactions between particles are represented by the complete quaternionic flag variety $Sp(n)/\bigotimes_1^n Sp(1)$.

INTRODUCTION

It is common practice in the sciences to formulate theories for isolated systems. These are systems in which everything of interest to the theorist happens within the system and all other interactions that the system might be subjected to are dismissed as negligible or irrelevant. For example, in discussing the structure of an atom one typically considers only the interactions within the atom and ignores the environment. That information is deemed to be irrelevant to the atomic structure at the chosen level of description. However, if the properties of, say, a crystal containing this atom are of interest, then the interactions between the atoms in the crystal become the focus, but interaction between the crystal and its environment are ignored. In all problems of this sort it is useful, indeed imperative, to consider the material system of interest to be decoupled from all external influences, thereby permitting only those interactions that arise from within the system to be considered. Such an isolated system could be of any size – an isolated system need not be small. The solar system is considered to be isolated when calculating the motions of planets; one may even assert that the observable universe is an isolated system. This is not so grandiose – it is merely the statement that all observable phenomena originate within the universe and do not come from elsewhere.

At the core of its definition, an isolated system has no interactions with anything other than itself. Consider a system, as simple or complex as you like, embedded in an otherwise empty space of any dimension. The absence of interactions with any external agent is assured if the isolated system is the only object that is embedded in this space. An isolated system cannot be observed, but an observer can be a part of an isolated system and can make observations on subsystems. Now, a question such as: "Is the system translating?" must be answered in the negative, as no observation can be made from the system to determine if it is translating – there are no reference points external to the system with respect to which an observer might verify rectilinear motion. The only motions that are verifiable to an observer within an isolated system occur when parts of the system move relative to one another. This is the essential content of special relativity. In the treatment to follow, this point of view will be expanded to describe complicated motions of systems consisting of several parts, where everything moves and rotates at once.

It is natural to look to a group to describe the simultaneous relative displacements of

many objects. To do this, the group is required to take an *active* role in the theoretical description. Suppose that there exists a group G that acts naturally and transitively on the configuration space, x , of an isolated system consisting of several parts, these parts being described by components of the space x . This configuration space can be as simple or complex as imagination allows, but by insisting on a transitive action, every configuration of the system is accessible from a standard configuration by the action of an element of the group. Whether or not all configurations that the group generates are physically acceptable depends on the nature of the objects and how the group is structured.

BASIC CONSTRUCTION

Two slightly different arguments will be presented for arriving at the structure declared in the abstract. The first is informal, the second more formal. In the first construction, it is useful to think of an isolated system as a set of objects in states of motion that are induced by natural laws. Without knowing what these laws might be, what can one infer about them by simply describing the relative configurations of the objects? This description must focus on configurations and not on the manifold in which they might be embedded. For example, if the objects are countable and separable material points, they could be singularities in an otherwise smooth manifold.

First Construction

Suppose that a group G_a acts transitively on the configuration space of an isolated system S_a . If one such group exists, there can be another G_b for S_b , treating this second system as isolated from the first. Now suppose the systems are combined into $S_{ab} = S_{a \cup b}$. (This operation should be understood as a union of sets.) The group $G_{ab} = G_{a \cup b}$ acts on the combined system, and in so doing encompasses all its relative motions. The product, $G_a \times G_b$, acts on the independent individual systems, each enjoying internal motions as if the other were not present. The coset $G_{ab}/G_a \times G_b$ therefore moves the parts of S_a because of the presence of S_b and *vice versa*. This obviously means that there is an interaction between the two systems – the coset $G_{ab}/G_a \times G_b$ *represents* these interactions. Clearly the argument can be extended to a system consisting of several subsystems, S_a, S_b, \dots, S_z , such that the

interactions between all pairs of subsystems are described by the coset $G_{ab\dots z}/G_a \times G_b \times \dots \times G_z$. Goldstone bosons have a G/H interpretation[1], and the above construction is the basis for a generalization to many-body systems.

The construction can be used for a system of n particles. Suppose that these are fermions, and that the properties of each are described by functions $\psi_k(x)$, $1 \leq k \leq n$. The group G_n is assumed to have a natural action on the coordinates of the configuration space x that is both continuous and transitive, as in representation theory where a representation A_g of $g \in G$ acts on functions $f(x)$ by $A_g f(x) = f(g^{-1}x)$. By the construction in the previous paragraph, we are required to define a group, G_1 , that acts on each fermion as if it is an isolated system, and such that a coset describes the interaction between this particle and its surroundings. What known property or properties of an individual fermion, when considered as an isolated object, can be defined solely within the context of a continuous (Lie) group? The only non-trivial possibility is *spin*. On referring back to the discussion of isolated systems, one has to conclude that charge and mass are not intrinsic properties of isolated particles, but instead require an interaction to define or quantify them. In the usual picture, a charge or mass generates a field, but to measure a field requires the presence of a test particle outside the system. Inertial mass can only be defined if acceleration relative to other objects is verifiable, *i.e.*, an external force is acting, and gravitational mass has meaning only where an external test mass detects its presence. While this argument may seem to be too classical, mass and charge are classical concepts, and the language is appropriate for the concepts. Spin is the highly non-classical answer to the question posed above: our group G_1 acting on an isolated fermion can only be $Spin(3) \sim SU(2) \sim Sp(1)$.

The structure that has been built is a principal bundle[2, 3], but it is also a flag variety[4, 6, 7]. The spin degrees of freedom are contained in a G_1 fiber sitting over each fermion. The interactions live on the base space G/H , with $H := \bigotimes_1^n G_1$. Because the *intrinsic* group G_1 describing spin is in the fiber direction, orthogonal to the base space, it cannot be defined using only coordinates of the base space. A single material particle will thus be described by a vector $V_k(\mathbb{F})$ having a number of components, k , to be determined by the algebraic field \mathbb{F} selected to define the spin degree of freedom.

Second Construction

Suppose that we are given a vector

$$\Psi_n(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_n(x) \end{bmatrix}$$

of single particle states for n fermions that are functions of the coordinates of a space x , these particles comprising an isolated system. Suppose further that this vector valued function is square integrable over x : $\int \Psi_n^*(x) \Psi_n(x) dx$ exists. The integral is invariant to $\Psi_n(x) \rightarrow U(n, \mathbb{F}) \Psi_n(x)$ where $U_n(\mathbb{F})$ is a unitary matrix over a field \mathbb{F} chosen to be compatible with the physical properties ascribed to the $\psi_j(x)$. A unitary transformation is permitted by the fundamental principle of quantum mechanics stating that linear combinations of wave functions constitute allowed states.

Now suppose that one of the particles is isolated from the rest. That is, let the group of isomorphisms of the vector space be $U_1(\mathbb{F}) \times U_{n-1}(\mathbb{F})$. In isolating one particle from the rest there is no interaction whatsoever between the particle and the remaining $n - 1$ particles. Stated in the reverse sense, the coset $U_n(\mathbb{F}) / U_1(\mathbb{F}) \times U_{n-1}(\mathbb{F})$ represents the interaction between the wave function for one particle and that of the remaining particles. This construction can be continued recursively to yield $U_n(\mathbb{F}) / \bigotimes_1^n U_1(\mathbb{F})$. The isolated fermions have spin, which is sufficient to identify $U_1(\mathbb{F})$ as above. Without explicit spin, the gauge group for each function is the scalar Berry phase[8] $U_1 = U_1(\mathbb{C})$, which gives the structure $U(n, \mathbb{C}) / U(1, \mathbb{C})^n$ that has been previously encountered[4, 5, 8].

Specifying the Field

Which family – orthogonal, unitary, or symplectic – should we choose for $G_1 \sim U_1$, with corresponding algebraic fields \mathbb{R} , \mathbb{C} , or \mathbb{H} ? To decide this we look at the parent group, G_n , and try the simplest answer: we want either $SO(3n) \sim Spin(3n)$ or $SU(2n)$ or $Sp(n)$ to conform homogeneously to the lowest order subgroups, respectively, and to be smoothly

applicable to systems of any n . The dimensions of the groups are[3]

$$\dim [SO(3n)] = 3n(3n - 1)/2$$

$$\dim [SU(2n)] = (2n + 1)(2n - 1)$$

$$\dim [Sp(n)] = n(2n + 1).$$

Now consider a system consisting of a single particle and its surroundings. As above, the interactions between the particle and its surroundings will be described by the coset $G_n/G_1 \times G_{n-1}$. The dimensions of the coset spaces are[3]

$$\dim [SO(3n)/SO(3) \times SO(3n - 3)] = 9(n - 1)$$

$$\dim \{SU(2n)/S[U(2) \times U(2n - 2)]\} = 8(n - 1)$$

$$\dim [Sp(n)/Sp(1) \times Sp(n - 1)] = 4(n - 1)$$

(The subgroup for the unitary case is chosen as $S[U(2) \times U(2n - 2)]$ rather than $SU(2) \times SU(2n - 2)$ so as to eliminate one annoying degree of freedom.) That is, the interaction between the subject particle and each particle in the surroundings has dimension nine, eight, or four. Pairwise interactions with eight or nine degrees of freedom pose a significant interpretation problem, but those with four are natural. The only realistic choice is the symplectic group. We have arrived at the coset space $Sp(n)/\bigotimes_1^n Sp(1) \sim Sp(n)/Sp(1)^n$ to describe interactions between particles. This is a complete quaternionic flag manifold – it has an extremely rich mathematical structure.

There is another reason for preferring the quaternion division algebra rather than the real or complex – the quaternions provide algebraic rigidity to space-time. To clarify this concept, consider the real case: there is no apparent reason for nature to prefer $SO(3n)/SO(3) \times SO(3n - 3)$ rather than, say, $SO(2m)/SO(2) \times SO(2m - 2)$ or $SO(4p)/SO(4) \times SO(4p - 4)$, where $3n = 2m = 4p$ for the sake of this argument. If we had chosen either the orthogonal or unitary structure we would have to outlaw these types of spaces with an additional physical law. The quaternions provide a natural, self-contained, space-time structure without the need for any additional assumptions.

This is an appropriate place to review the structure that has been built. The first premise is that one can define an isolated system as a collection of objects, subject to natural laws, that in being isolated has no interactions whatsoever with anything outside itself. In particular, an isolated system cannot interact with an external observer, but an observer can

be part of an isolated system. The second assertion is that the various configurations of the objects in an isolated system can be described by the action of a group on the configuration space of these objects. Third, objects may be elementary particles whose sole intrinsic property requires a vector-valued function, and not a scalar, to describe it. The fourth and final assumption is that pairwise interactions between elementary particles require only four independent components to describe them fully.

The symplectic group is also denoted by $U(n, \mathbb{H})$, *i.e.*, it is a unitary group over the quaternion ring \mathbb{H} . The group is both symplectic and unitary: $Sp(n) \sim Sp(2n, \mathbb{C}) \cap U(2n, \mathbb{C})$ [9]. It will often be convenient to shorten notation, as above, to $Sp_k := Sp(k)$. The group $Sp(n)$ is compact and has no obvious connection to relativity, nor to the Lorentz or Poincaré group. However, systems and subsystems move relative to one another under the action of this group, and that is sufficient to realize all accessible physical states. Connections to the Lorentz group and deSitter and Anti-deSitter (AdS) spaces will be established later.

COMPOSITE SYSTEMS

If individual subsystems can be distinguished from other subsystems they are at least partially isolated from one another. This is precisely the structure considered in the First Construction. A partially isolated system consisting of k fermions is represented by a vector space $V_k(\mathbb{H})$ that is invariant to the action of $Sp(k)$. Several subsystems can comprise a larger system that is invariant to $Sp(n)$, and the interactions between the subsystems can be described by the partial flag variety $Sp(n)/[Sp(k_1) \times Sp(k_2) \times \cdots \times Sp(k_\tau)]$ such that $\sum_{j=1}^\tau k_j = n$. This flexibility of group representations provides a mechanism for transitioning from indistinguishable particles within a subsystem to preserving the individuality of subsystems. Representations of $Sp(n)$ that are induced by representations of the "maximal factor group" $\bigotimes_{j=1}^\tau Sp(k_j) \sim H_K$, (K is a partition of n) are the essence of this theory. In the mathematical literature, flag manifolds are often constructed by moding out by a maximal parabolic subgroup, G/\mathcal{P} [10]. The choice of factor group made here provides a more cogent physical interpretation than this alternative representation, which is not to say that $Sp(n)/\mathcal{P}$ might not be well suited to understanding some aspects of the theory or doing computations.

THE ACTION OF $SL(2, \mathbb{C}) / \{\pm 1\} \sim SO(3, 1)$

A thorough study of the relation between quaternion flag varieties and traditional field theories will require considerable effort. As a start, one simply notes that in generally accepted field theories all interactions between a particle and its surroundings take place in a four-dimensional space-time manifold with Lorentz signature rather than in this pairwise additive $4n$ -dimensional setting. The special theory of relativity is built on Lorentz transformations describing the relative motion of *pairs* of objects. In the theory under construction, a pairwise interaction between particles a and b is conveyed by quaternionic matrix elements q_{ab} and q_{ba} .

The largest group that preserves the norm of a single quaternion (and its conjugate) in the $q \sim \mathbb{R} \times SU(2, \mathbb{C}) \in GL(2, \mathbb{C})$ representation, where the norm is the determinant, is $u \in SL(2, \mathbb{C})$ acting by a similarity transformation $u : q \rightarrow u^* q u$. A map between the quaternion representation and space-time with a Lorentz signature exists, and this is encompassed by the well-known group homomorphism of the title of this section. A Wick rotation of the identity component of the quaternion will provide this identification, but there are additional correspondences between this theory and relativity that will become apparent as the subject is developed. For now, it is suggested that the reader not form a strong opinion as to the nature of the identity component of our quaternions. The relation between Galilean time and the identity components of the quaternions is not immediately obvious, precisely because each quaternion has an identity component. The theory under development might provoke renewed thinking about the nature of temporal coordinates.

BOTTOM UP CONSTRUCTION AND GROUP ACTION

The complete quaternionic flag variety $Sp(n)/[Sp(1)]^n$ is difficult to handle directly, so a (local) parameterization via the factorization

$$Sp_n/[Sp(1)]^n = [Sp_n/Sp_1 \times Sp_{n-1}] \times [Sp_{n-1}/Sp_1 \times Sp_{n-2}] \times \cdots \times [Sp_2/Sp_1 \times Sp_1]$$

enables one to build up solutions by solving the smallest problems first. This representation is a product of hyper-Kähler manifolds[11, 12]. At the bottom of these nested coset spaces is $Sp(2)/Sp(1) \times Sp(1)$, which is the space of solutions of the Yang-Mills functional[13].

Using standard group isomorphisms this space is alternatively represented by

$$Sp(2)/Sp(1) \times Sp(1) = SO(5)/SO(3) \times SO(3) = SO(5)/SO(4) = S^4$$

i.e., the four-sphere[13, 14]. The four-sphere will be discussed at greater length later.

The action of the respective groups on their coset spaces is given by linear fractional transformations. To show this, embed the coset space $Sp_{k+1}/Sp_1 \times Sp_k$ in Sp_n , $n > k + 1$, and let it be parameterized by the elements

$$\exp \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\xi^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1 - ZZ^*)^{1/2} & Z \\ 0 & -Z^* & (1 - Z^*Z)^{1/2} \end{bmatrix}$$

derived from the Lie algebra[15]. Here ξ is a k -dimensional vector over the quaternions, and $Z = (\xi\xi^*)^{-1/2} \left[\sin(\xi\xi^*)^{1/2} \right] \xi = \xi(\xi^*\xi)^{-1/2} \sin(\xi^*\xi)^{1/2}$, the latter being defined by the formal power series. The conjugate transpose of a quaternion vector x is denoted by x^* (components may be represented in either the algebraic \mathbb{H} or matrix $\mathbb{R} \times SU(2)$ forms). (For $k + 1 < n$ the above representation is embedded in the $n \times n$ larger matrix as shown; in the sequel this embedding will be understood and the padding will be omitted.)

An element $g \in Sp_{k+1}$ expressed in a conforming partitioning is

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \quad g^{-1} = g^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix},$$

and this acts by

$$\begin{aligned} gxH &= yH \\ gxH &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (1 - ZZ^*)^{1/2} & Z \\ -Z^* & (1 - Z^*Z)^{1/2} \end{bmatrix} H \\ yH &= \begin{bmatrix} A(1 - ZZ^*)^{1/2} - BZ^* & AZ + B(1 - Z^*Z)^{1/2} \\ C(1 - ZZ^*)^{1/2} - DZ^* & CZ + D(1 - Z^*Z)^{1/2} \end{bmatrix} H \end{aligned}$$

where $H \sim Sp_1 \times Sp_k$. This construction works for a larger class of coset spaces than is implied here; Z might just as well belong to $Sp(k)/Sp(j) \times Sp(k-j)$. At several places ahead this larger coset space will be intended in the development.

Physically what is happening is that the system, the $Sp(j)$ part, and the surroundings, the $Sp(k-j)$ part, may experience an arbitrary change in their spin/internal states as a

result of the action of g , and this change is conveyed by the subgroup H . We want to know how g acts on the coset space, and this requires that the action of H be eliminated by taking ratios. (This construction will become clearer in the discussion of induced representations to follow.) The action of $G \sim Sp(k)$ on $G/H \sim Sp(k)/Sp(j) \times Sp(k-j)$, ($g : Q \rightarrow P$), is more succinctly represented by

$$P = (AQ + B)(CQ + D)^{-1} = (-QB^* + A^*)^{-1}(QD^* - C^*) \quad (1)$$

with $Q = Z(1 - Z^*Z)^{-1/2} = (1 - ZZ^*)^{-1/2}Z$. Given this action it is seen that the origin of the inhomogeneous space is mapped to $g : O \rightarrow BD^{-1} = -(A^*)^{-1}C^*$, which is useful for some computations. Clearly the coset space has the appearance of being non-compact, as a point satisfying $CQ + D = 0$ or $A^* - QB^* = 0$ is mapped to infinity.

METRIC AND CURVATURE

This section is standard material and only the results will be presented[16]. The invariant line element on $Sp(k)/Sp(j) \times Sp(k-j)$ is given by

$$ds^2 = \text{tr} [(1 + QQ^*)^{-1} dQ (1 + Q^*Q)^{-1} dQ^*] \quad (2)$$

$$= \text{tr} [(1 + QQ^*)^{-1} dQ dQ^* - (1 + QQ^*)^{-1} dQQ^* (1 + QQ^*)^{-1} Q dQ^*] \quad (3)$$

where the second version follows from $(1 + Q^*Q)^{-1} = 1 - Q^* (1 + QQ^*)^{-1} Q$; Q is a $j \times (k-j)$ matrix of quaternions. Calculation of the Maurer-Cartan form ([2],[17]) for this general case gives the curvature form as

$$\Omega = \begin{bmatrix} h_j^* & 0 \\ 0 & h_{k-j}^* \end{bmatrix} \begin{bmatrix} dQ \wedge dQ^* & 0 \\ 0 & dQ^* \wedge dQ \end{bmatrix} \begin{bmatrix} h_j & 0 \\ 0 & h_{k-j} \end{bmatrix}. \quad (4)$$

Here $dQ = (1 + QQ^*)^{-1/2} dQ (1 + Q^*Q)^{-1/2}$, $h_j \in Sp_j$ and $h_{k-j} \in Sp_{k-j}$. This identifies the $Sp_k/Sp_j \times Sp_{k-j}$ coset space as an Einstein space; for $k = 1$ the space is also a K3 surface[18]. The form encompasses both self-dual and anti-self-dual sectors for $n = 2$ [13].

EQUATION OF MOTION AND LIE ALGEBRA

On returning to the wave function description, let

$$\Psi(xH) = \begin{bmatrix} \psi_\alpha(xH) \\ \psi_a(xH) \end{bmatrix}$$

be a square-integrable vector-valued function that is compatible with the matrix representation of the group that we have been working with. This is a vector bundle associated to the principle bundle[2, 19]. It is useful to think of $\psi_\alpha(xH)$ as the wave function of the system, and $\psi_a(xH)$ as that of the surroundings. (The reason for the peculiar indexing will be apparent shortly.) The most natural equation of motion that is consistent with quantum theory and our group is provided by a one-parameter local group action with Lie algebra \mathfrak{g} , so that

$$\partial\Psi/\partial t = \mathfrak{g}\Psi \quad (5)$$

where \mathfrak{g} consists of the infinitesimal generators of the Lie algebra, *i.e.*, the Lie derivative. The parameter t is safely identified with Galilean time. (We are using natural units with $\hbar = c = 1$. Furthermore, skew-symmetry of the Lie algebra, with concomitant suppression of $\sqrt{-1}$, is preferred over the Hermitean option so as to avoid clutter.) This equation might be derived from a conservation principle; the total time derivative of a p -form over a manifold is given by an equation of this type[20].

We now need to fix a representation for the quaternions so as to present the Lie algebra. The standard basis for the quaternions consists of $\{\mathbf{e}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, with

$$\mathbf{e} = \mathbf{1}; \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{e} = -\mathbf{1}; \quad \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}$$

A quaternion q will be written $q := w\mathbf{e} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, with conjugate $q^* := w\mathbf{e} - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. The norm $|q|$ of q is defined by $|q|^2 = qq^* = q^*q = (w^2 + x^2 + y^2 + z^2)\mathbf{e}$. The derivative is most naturally defined such that $dq/dq = 1$, which implies that

$$d/dq = \frac{1}{4}(\mathbf{e}\partial/\partial w - \mathbf{i}\partial/\partial x - \mathbf{j}\partial/\partial y - \mathbf{k}\partial/\partial z).$$

Choose a representation of a quaternion in a basis of Pauli matrices (modulo $i = \sqrt{-1}$) as

$$q = \begin{bmatrix} w + iz & x + iy \\ -(x - iy) & w - iz \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 \\ -\bar{\rho}_2 & \bar{\rho}_1 \end{bmatrix}$$

and

$$q^* = \bar{q}' = \begin{bmatrix} \bar{\rho}_1 & \bar{\rho}_2 \\ -\rho_2 & \rho_1 \end{bmatrix}' = \begin{bmatrix} \bar{\rho}_1 & -\rho_2 \\ \bar{\rho}_2 & \rho_1 \end{bmatrix} \quad (6)$$

where \bar{r} is the complex conjugate and r' is the transpose of the matrix r . The differential

operator in the matrix representation is

$$d/dq = \frac{1}{2} \begin{bmatrix} \partial/\partial\rho_1 & -\partial/\partial\bar{\rho}_2 \\ \partial/\partial\rho_2 & \partial/\partial\bar{\rho}_1 \end{bmatrix} \quad (7)$$

This choice for d/dq gives $dq/dq = 1$ and $d\bar{q}/dq = 0$, but $dq^*/dq = -1/2$. (It is sometimes easier to work with the algebraic representation, \mathbb{H} , than with the matrix representation in explicit calculations. However, quaternionic differentiations are intrinsically non-commutative and can be hazardous – for calculation of commutation relations, for example, it is safer to use the matrix representation.)

An important automorphism is provided by $J_1 = \mathbf{i}$ acting by conjugation $J_1 \cdot q \rightarrow J_1^{-1}qJ_1 = \bar{q}$ and $J_1^{-1}\partial J_1 = \bar{\partial}$. Define $J_k := \mathbf{1}_k \otimes J_1$ where $\mathbf{1}_k$ is the identity matrix of rank k and \otimes is the direct product of matrices. Conjugation of a k -dimensional quaternion-valued vector Q is provided by $\bar{Q} = J_1^{-1}QJ_1$, so that $Q^* = J_k Q' J_1^{-1}$. These relations are very helpful in proving the commutation relations below.

The infinitesimal generators of the Lie algebra, $\mathfrak{g} \in \mathfrak{sp}_n$, that are parameterized by the coordinates of $Sp_n/Sp_k \times Sp_{n-k}$ are best written using different fonts for row and column indices, corresponding respectively to the \mathfrak{sp}_k and \mathfrak{sp}_{n-k} subalgebras. The summation convention is used in the next several equations defining the generators and their commutation relations:

$$h_{\alpha\beta} = q_{\alpha b}\partial_{\beta b} - \bar{q}_{\beta b}\bar{\partial}_{\alpha b} \quad (8a)$$

$$h_{ab} = q_{\mu a}\partial_{\mu b} - \bar{q}_{\mu b}\bar{\partial}_{\mu a} \quad (8b)$$

$$p_{\alpha a} = \bar{\partial}_{\alpha a} + q_{\alpha b}q_{\mu a}\partial_{\mu b} \quad (8c)$$

$$= (\delta_{\alpha\beta} + q_{\alpha b}\bar{q}_{\beta b})\bar{\partial}_{\beta a} + q_{\alpha b}h_{ab} \quad (8d)$$

$$= (\delta_{ab} + q_{\mu a}\bar{q}_{\mu b})\bar{\partial}_{\alpha b} + q_{\mu a}h_{\alpha\mu}. \quad (8e)$$

Here $q_{\alpha a}$ is one component of a $k \times (n-k)$ matrix of quaternions and $\partial_{\alpha a}$ is the corresponding partial derivative, $\partial_{\alpha a} = \partial/\partial q_{\alpha a}$. (The indexing in these equations is necessarily clumsy to keep all derivatives to the right of coordinates. They are simpler to write with single symbol matrix notation, *e.g.*, $h_{\alpha\beta} = [Q\partial' - (Q\partial')^*]_{\alpha\beta}$.) A straightforward but tedious calculation

shows that these generators follow the canonical commutation relations[3]

$$\begin{aligned}
[h_{\alpha\beta}, h_{\mu\nu}] &= \delta_{\beta\mu} h_{\alpha\nu} - \delta_{\alpha\nu} h_{\mu\beta} \\
[h_{ab}, h_{cd}] &= \delta_{bc} h_{ad} - \delta_{ad} h_{cb} \\
[h_{\alpha\beta}, h_{ab}] &= 0 \\
[h_{\alpha\beta}, p_{\mu a}] &= \delta_{\beta\mu} p_{\alpha a} \\
[h_{\alpha\beta}, \bar{p}_{\mu a}] &= -\delta_{\alpha\mu} \bar{p}_{\beta a} \\
[h_{ab}, p_{\mu c}] &= \delta_{bc} p_{\mu a} \\
[h_{ab}, \bar{p}_{\mu c}] &= -\delta_{ac} \bar{p}_{\mu a} \\
[p_{\alpha a}, p_{\beta b}] &= 0 \\
[p_{\alpha a}, \bar{p}_{\beta b}] &= -\delta_{\alpha\beta} h_{ab} - \delta_{ab} h_{\alpha\beta}.
\end{aligned}$$

It is convenient to change notation for the wave functions in the following to clarify content. Define a vector $(v, V)'$ with components in \mathbb{H} , where $\dim(v) = k$ and $\dim(V) = n - k$. (An aside at this point is in order. One might take the state vector for a single particle to lie in $(\phi_1, \phi_2) \in \mathbb{C}^2$. On constructing this state vector, there exists a "conjugate" state $(-\bar{\phi}_2, \bar{\phi}_1)$ that is immediately adjoined. One is naturally confronted with the quaternionic representation by the structure of the group.) The Lie algebra acts on this vector space by

$$\begin{bmatrix} \mathfrak{h}_v & -\mathfrak{p} \\ \mathfrak{p}^* & \mathfrak{h}_V \end{bmatrix} \begin{bmatrix} v \\ V \end{bmatrix} = \begin{bmatrix} \mathfrak{h}_v v - \mathfrak{p} V \\ \mathfrak{p}^* v + \mathfrak{h}_V V \end{bmatrix}, \quad (9)$$

where $\mathfrak{h}_v \in Sp_k$ and $\mathfrak{h}_V \in Sp_{n-k}$. The \mathfrak{h}_v and \mathfrak{h}_V operators "rotate" the system (Sp_k part) and the surroundings (Sp_{n-k} part), respectively, while the \mathfrak{p} and \mathfrak{p}^* operators induce transitions between them.

That the \mathfrak{p} and \mathfrak{p}^* operators induce transitions is easily seen using standard operations. Suspend the summation convention, and let V_a be an eigenvector with eigenvalue n_a of h_a in the Cartan subalgebra of Sp_{n-k} . The action of $p_{\alpha a}$ on this eigenvector is easily deduced from

$$h_a p_{\alpha a} V_a = p_{\alpha a} h_a V_a + [h_a, p_{\alpha a}] V_a = n_a p_{\alpha a} V_a + p_{\alpha a} V_a = (n_a + 1) p_{\alpha a} V_a \quad (10)$$

with use of the commutation relations. Simultaneously, the conjugate $\bar{p}_{\alpha a}$ acts on v_α , an eigenvector of $h_\alpha \in Sp_k$ with eigenvalue n_α . This is a lowering operator, as is seen from

$$h_\alpha \bar{p}_{\alpha a} v_\alpha = \bar{p}_{\alpha a} h_\alpha v_\alpha + [h_\alpha, \bar{p}_{\alpha a}] v_\alpha = n_\alpha \bar{p}_{\alpha a} v_\alpha - \bar{p}_{\alpha a} v_\alpha = (n_\alpha - 1) \bar{p}_{\alpha a} v_\alpha. \quad (11)$$

The location of raising and lowering operators in this construction is arbitrary as are the signs; interchanging p and \bar{p} reverses the direction of the transition, raising v_α and lowering V_a . This demonstrates that the algebra encompasses exchange of excitations between system and surroundings in either direction. Energy exchange will likely be more interesting to study in the context of the group, where topological considerations will be important. The theory has the right ingredients for application of the "bubbling off" theorem[14], and a study of excitation transfer in that context might provide insight into the relation between localization of curvature and conical intersections.

The Laplace-Beltrami (LB) operator, Δ , associated to these generators is the trace of the square of the matrix of generators. Thus

$$\Delta = \text{tr}(\mathfrak{h}_1 \mathfrak{h}_1^* + \mathfrak{p} \mathfrak{p}^*) + \text{tr}(\mathfrak{h}_k \mathfrak{h}_k^* + \mathfrak{p}^* \mathfrak{p}).$$

This is only one among many of the composite operators that can be formed from the generators of the Lie algebra.

On returning to eq. (5) one notes that the eigenvalues of a symplectic matrix are pure imaginary[9, 10] and occur in conjugate pairs, which recommends that we look for eigenvector solutions of the equation with $\Psi(t, xH) = \Psi(xH) \exp(\theta t \mathbf{k})$. The equation to be solved is thus

$$\theta \begin{bmatrix} \psi_\alpha(xH) \\ \psi_a(xH) \end{bmatrix} \mathbf{k} = \begin{bmatrix} \mathfrak{h}_\alpha & -\mathfrak{p} \\ \mathfrak{p}^* & \mathfrak{h}_a \end{bmatrix} \begin{bmatrix} \psi_\alpha(xH) \\ \psi_a(xH) \end{bmatrix}. \quad (12)$$

The eigenvalues θ lie in the maximal torus and are not simply identified as energies. There is a further remarkable feature of eq. (5), perhaps best described in the time dependent form (with t implicit in functions). Writing out the right hand side one has

$$\begin{bmatrix} \partial \phi_\alpha(xH) / \partial t \\ \partial \phi_a(xH) / \partial t \end{bmatrix} = \begin{bmatrix} \mathfrak{h}_\alpha \phi_\alpha(xH) - \mathfrak{p} \phi_a(xH) \\ \mathfrak{p}^* \phi_\alpha(xH) + \mathfrak{h}_a \phi_a(xH) \end{bmatrix}. \quad (13)$$

At an instant of observation, as registered by the change in the wave function $\partial \phi_a(xH) / \partial t$ of the *surroundings*, the system $\phi_\alpha(xH)$ reports its *present* state via the operator \mathfrak{p}^* . However, since the system is in contact with its surroundings *via* the $\mathfrak{p} \phi_a(xH)$ term, its state will evolve and the next observation will find the system in a different state. (While the equation is written for a single particle, the argument works for many-body systems as well.) Whether the change of state is large or small depends on the strength of the coupling, and that is a

problem for another time. In any event, the physical content of eq. (13) is that we infer the properties of matter through its interaction with experimental apparatus (or by personal observation).

The \mathfrak{h}_α operator in eqs. (12,13) restricted to a single particle is a spin operator that *is* a function of the coordinates in the base space. In the fiber bundle picture what is happening is that the Lie algebra transports the system in a tangent plane, and this projects onto a motion along the $Sp(1)$ fiber because the base space is not flat.

The block diagonal \mathfrak{h} angular momentum operators have a clear interpretation as internal degrees of freedom. These internal motions of individual subsystems are conveyed between subsystems by \mathfrak{p} as is clear in eqs. (8d,8e). As we have seen, the \mathfrak{p} operators convey interactions between different systems and provide the deepest insights into the theory. To develop this topic while avoiding a plethora of indices, it is useful to look at individual terms in eqs. (10). Dropping inessential indices gives

$$\bar{\partial}_{\alpha\alpha} = \partial/\partial\bar{q} = \frac{1}{4}(\mathbf{e}\partial_0 + \mathbf{i}\partial_1 + \mathbf{j}\partial_2 + \mathbf{k}\partial_3) = \frac{1}{4}(\mathbf{e}\partial_0 + \nabla)$$

where, on the right, $\partial_m = \partial/\partial x_m$. This derivative acts on a quaternion $A = 4(\mathbf{e}A_0 + \mathbf{i}A_1 + \mathbf{j}A_2 + \mathbf{k}A_3) = 4(\mathbf{e}A_0 + \mathbf{A})$ to give

$$\partial A/\partial\bar{q} = (A_{0,0} - \nabla \cdot \mathbf{A})\mathbf{e} + [\mathbf{A}_{,0} + \nabla A_0 + \nabla \times \mathbf{A}] \quad (14)$$

which are the components of the electromagnetic (EM) field written in the classical and traditional space-time form. The scalar term is not an arbitrary gauge; the $Sp(1)$ fibres are the arbitrary gauge terms, as will be seen later in the context of induced representations. It is probable that the identity component of this derivative conveys a great deal of information, but no further comments are offered here.

As $p_{\alpha\alpha}$ acts on the vector field $\psi_a(\mathbf{q})$ for a single particle, this term in the Lie derivative projects the electromagnetic field of the surroundings onto the particle. Since the fermionic states will be quantized, the EM interactions inherit this quantization. Simultaneously, the corresponding term in $\bar{p}_{\alpha\alpha}$ projects the field of the particle onto the surroundings. The derivative with respect to q rather than \bar{q} changes the signs of terms in eq. (14), and modulo an overall sign is equivalent to reversing the sign of the identity component of the derivative, which is acting as the temporal coordinate. Since the role of time in classical EM is conveyed by the identity component of our quaternions, each interaction comes equipped with its own

time coordinate. The implications of this are best left to others. As a row in \mathbf{p} acts it accumulates the EM fields of the surroundings to act on a single particle, which provides a clear path to transition from microscopic to macroscopic field behavior.

This discussion also enables us to finally interpret the coordinates in the group. By identifying eq. (14) with the components of the EM field, the coordinates in the q_{bb} have to be identified as the components of pairwise distances. Since the wave functions depend on the whole set of pairwise distances, this extensive set of coordinates allows for states having a richer internal structure than can be captured by the exclusion principle operating in \mathbb{R}^4 .

The $(\delta_{\alpha\beta} + q_{\alpha b}\bar{q}_{\beta b})$ and $(\delta_{ab} + \bar{q}_{\mu a}q_{\mu b})$ terms in eqs. (8d-8e) reduce to the identity for small $|q|$ and otherwise convey the non-euclidean character of the space. The $\Sigma_b q_{\alpha b}h_{ab}$ and $\Sigma_\mu q_{\mu a}h_{\alpha\mu}$ terms are analogous to the time derivative of a torque, the implications of which are left to the reader.

THE ALTERNATIVE REPRESENTATION OF $Sp(n)$

To this point the theory has been developed in the representation $U(n, \mathbb{H})$. Given the fact that quaternions are not commutative and have been sidelined since the time of Gibbs, some may find it more comfortable to work in the $Sp(2n, \mathbb{C})$ version. The conjugation operation $q^* = J_1^{-1}q'J_1$ provides just what is needed to map between the two representations[21]. For $g \in U(n, \mathbb{H})$ we have $g^*g = J_n^{-1}g'J_ng = 1$, so that $g'J_ng = J_n$, where $J_n := \mathbf{1}_n \otimes J_1$ as before. Now, there exists a permutation \mathcal{P} such that $\mathcal{P} : (\mathbf{1}_n \otimes J_1) \rightarrow J_1 \otimes \mathbf{1}_n$, and acting on g gives a permuted form $\mathcal{P} : g \rightarrow \mathcal{G}$. For $\mathcal{J} = J_1 \otimes \mathbf{1}_n$, this gives $\mathcal{G}'\mathcal{J}\mathcal{G} = \mathcal{J}$, which is the standard definition of the symplectic group over the complex numbers [$\mathcal{G} \in Sp(2n, \mathbb{C})$]; the group preserves a complex skew-symmetric bilinear form. But the group is also unitary, *i.e.*, $Sp(n) \sim Sp(2n, \mathbb{C}) \cap U(2n, \mathbb{C})$, as noted above, so that we also have $\mathcal{G}^*\mathcal{G} = 1$. These two properties yield the Lie algebra in this representation in the form

$$\mathfrak{g} \sim \begin{bmatrix} \mathfrak{a} & \mathfrak{b} \\ -\mathfrak{b}^* & -\mathfrak{a}' \end{bmatrix}; \quad \mathfrak{a}^* = -\mathfrak{a}; \quad \mathfrak{b}' = \mathfrak{b}$$

where \mathfrak{a} and \mathfrak{b} are complex matrices. Cosets in this representation are messier to work with than those in the quaternion basis, so this representation is not pursued further here. For example, the components of the $\mathfrak{sp}(1)$ fibres are split between the diagonal elements of \mathfrak{a}

and \mathfrak{b} . In any event, one should note that the \mathfrak{a} -sector of this representation, taken alone, is isomorphic to $U(n)$.

AVERAGING INTERACTIONS: THE LARGE N LIMIT

There has been some discussion in the literature of large n theories, and a few comments on this topic may be useful. Contact with modern field theories will at least require that the metric associated to the n -dimensional quaternion Q be simplified to one or a few quaternion dimensions. The most straightforward way to do this is by averaging. For the $Sp_{n+1}/Sp_1 \times Sp_n$ metric in eq. (2) one may introduce the averages

$$n \langle d\sigma d\sigma^* \rangle = dQdQ^*; \quad n \langle \sigma\sigma^* \rangle = n|\sigma|^2 = QQ^*; \quad n \langle \sigma d\sigma^* \rangle = QdQ^* \quad (15)$$

so that the metric can be written as

$$ds^2 = \text{tr}\{f(n|\sigma|^2) |\sigma|^{-2} [\langle d\sigma d\sigma^* \rangle - \langle \sigma d\sigma^* \rangle^* f(n|\sigma|^2) |\sigma|^{-2} \langle \sigma d\sigma^* \rangle]\} \quad (16)$$

$$\begin{aligned} &\approx \text{tr}\{(1 - n^{-1}|\sigma|^{-2}) |\sigma|^{-2} [\langle d\sigma d\sigma^* \rangle - \langle \sigma d\sigma^* \rangle^* |\sigma|^{-2} \langle \sigma d\sigma^* \rangle]\} \\ &\quad + \text{tr}\{(1 - n^{-1}|\sigma|^{-2}) |\sigma|^{-2} \langle \sigma d\sigma^* \rangle^* n^{-1}|\sigma|^{-4} \langle \sigma d\sigma^* \rangle\}. \end{aligned}$$

Here $f(x) = (1 + 1/x)^{-1}$; σ and $d\sigma$ are understood to be one-dimensional (resp. infinitesimal) quaternions that are defined in eq. (15). The second, approximate, version of the metric is valid for large n only. The metric on $Sp_2/Sp_1 \times Sp_1$ that we are trying to emulate is

$$ds^2 = \text{tr} \left[(1 + |q|^2)^{-2} dqdq^* \right]. \quad (17)$$

(Note that this is the metric for an instanton[13, 14]. The first term in the approximate version of eq. (16) can be made to vanish (at least at a point) by choosing the $\rho = \sigma/|\sigma|$ part of $\langle \sigma d\sigma^* \rangle$ such that $\langle \rho d\sigma^* \rangle \langle \rho d\sigma^* \rangle^* \approx \langle d\sigma d\sigma^* \rangle$. To first order in n^{-1} the metric then reduces to

$$ds^2 \approx n^{-1} \text{tr} [|\sigma|^{-2} \langle \sigma d\sigma^* \rangle |\sigma|^{-4} \langle \sigma d\sigma^* \rangle^*] \approx n^{-1} \text{tr} [|\sigma|^{-4} d\sigma d\sigma^*],$$

which seems to be as close as one can get to eq. (17) (corresponding to q large in that equation). This bears some resemblance to the large n limit of metrics considered by Maldacena[22]. More careful approximations for small systems embedded in large n surroundings will be very important to understanding how this theory works.

S^4 AND THE CONFORMAL GROUP

The isomorphism $Sp(2)/Sp(1) \times Sp(1) \sim SO(5)/SO(4) \sim S^4$ enables one to use real coordinates for calculations: the geometry is, of course, just

$$\sum_{i=0}^4 x_i^2 = 1.$$

$SO(5)$ acts by linear fractional transformations on the inhomogeneous coordinates $y_k = x_k/x_0, 1 \leq k \leq 4$, with $1 + yy' = 1/x_0^2 \geq 1$. Evaluation of the invariant metric is a standard calculation, yielding the line element in the y -coordinates as

$$ds^2 = (1 + yy')^{-1} dy (1 + y'y)^{-1} dy'.$$

The relation between the metrics on $Sp(2)/Sp(1) \times Sp(1)$ and $SO(5)/SO(4)$ is clear. The self-dual and anti-self-dual connections, Yang-Mills action, and instantons are more accessible from the quaternion version,[13],[14] but the real version exposes the relation with the de Sitter space.

The 4-sphere is the surface of the 5-ball, B^5 . Since our larger group (Sp_n) action admits inversions, one is encouraged to look also at the exterior of the sphere, obtained by inverting the ball through the S^4 surface. The extended Lorentz group $SO(1,5)$ acts on the ball (a de Sitter space), while the anti-de Sitter (AdS) space ($vv' - v_0^2 = 1 \Rightarrow xx' - 1 > 0; x = v/v_0$) is the inverted ball (here v is a 5-dimensional vector). The group $SO(1,5)$ is the conformal group of S^4 [22, 23]. The boundary, S^4 , of either B^5 or its inverse is approached in the limit $v_0 \rightarrow \infty$. No conjecture is offered as to what lives in the de Sitter or AdS space.

REPRESENTATIONS OF $Sp(k)$ FOR SMALL k

The primary reason for being interested in $Sp(k) \times [Sp(n)/Sp(k) \times Sp(n-k)]$ for small k is that these spaces presumably provide insight into elementary particle structure. The connection is stated as a conjecture:

- Leptons are represented by $Sp(1) \times [Sp(n)/Sp(1) \times Sp(n-1)]$
- Mesons by $Sp(2) \times [Sp(n)/Sp(2) \times Sp(n-2)]$
- Baryons by $Sp(3) \times [Sp(n)/Sp(3) \times Sp(n-3)]$

The relation between the rank of these spaces and topology is palpable. If this conjecture is correct, it is also clear why individual quarks are not isolable: A higher dimensional representation of, say, $Sp(3)$ might "come apart" (through its interactions with the surroundings) into pieces that are classified in $Sp(1)$, $Sp(2)$, or $Sp(3)$; *i.e.*, the decay products are either leptons, mesons or baryons. The "constituents" of a representation of $Sp(3)$ do not have an independent existence. However, the root space of $\mathfrak{sp}(2)$ is a subspace of that for $\mathfrak{sp}(3)$, which enables quark assignments to be made for $\mathfrak{sp}(2)$. On replacing Sp by SU , one gets the scalar version of the Grassmannians. Alternatively, since $SU(n) \subset Sp(n)$, the standard $SU(3)$ representation of baryons is accommodated in the symplectic versions. The product spaces $Sp(k) \times [Sp(n)/Sp(k) \times Sp(n-k)]$ are Steifel manifolds, isomorphic to $Sp(n)/Sp(n-k)$. The conjecture is phrased as it is to make the internal symmetries clearly identifiable. Note also that these Stiefelians describe only single particles and composites; multiple particles and composites require structures such as $Sp(n)/Sp(j) \times Sp(k) \times Sp(n-j-k)$, *etc.*

There is another aspect of representations that deserves mention. The maximal torus contains the eigenvalues θ_m in $\exp(\mathbf{k}\theta_m t) \in \Lambda$ that are solutions of eq.(5). The normalizer N of Λ contains diagonal matrices with elements of the form $\exp(\mathbf{k}\mathbf{p} \cdot \mathbf{x}_m)$, $1 \leq m \leq n$. Fix $y \in Sp(n)$ and let $g \in Sp(n)$ diagonalize y by $\Lambda = g^* y g = g^{-1} y g$. If $n \in N$ it follows that

$$n^* \Lambda n = \Lambda = (n^* g^* n)(n^* y n)(n^* g n)$$

signifying that the elements $\mathbf{y}_{mn} \in y$, with $\mathbf{y}_{mn} \in \mathbb{H}$ may contain arbitrary phase factors of the form $\exp[\mathbf{k}\mathbf{p} \cdot (\mathbf{x}_m - \mathbf{x}_n)]$. The vectors \mathbf{p} and \mathbf{x}_m are arbitrary.

Sp_2/Sp_1^2 ONCE AGAIN

The purpose of this section is to discuss a few additional aspects of the theory in the context of an illustrative calculation. To this end, it is useful to solve the smallest problem available, which will be taken to be the action of the Laplace-Beltrami operator on scalar functions on $S^4 \sim Sp_2/Sp_1^2$. The metric is $ds^2 = (1 + qq^*)^{-1} dq (1 + q^* q)^{-1} dq^* = (1 + |q|^2)^{-2} dq dq^*$. The substitution $q = \cot(\omega/2)u$, with $uu^* = 1$ gives $ds^2 = d\omega^2 \mathbf{1} + (\sin \omega)^2 \delta u \delta u^*$, where $\delta u = duu^* = -u du^* = -\delta u^*$. (An uninteresting numerical factor was dropped.) The reason for choosing $\cot(\omega/2)$ rather than $\tan(\omega/2)$ is that we want a singularity at the origin,

as will be explained shortly. A convenient parameterization of Sp_1 is

$$u = \exp(\mathbf{k}\beta/2) \exp(\mathbf{i}\alpha/2) \exp(\mathbf{k}\gamma/2) = bac$$

such that

$$\delta u = (1/2)b[d\alpha\mathbf{i} + d\beta\mathbf{k} + d\gamma \exp(\alpha\mathbf{i})\mathbf{k}]b^*$$

from which it follows that

$$ds^2 = 4d\omega^2 + \sin^2 \omega [d\alpha^2 + d\beta^2 + d\gamma^2 + 2(\cos \alpha)d\beta d\gamma].$$

The factor of four in this equation plays an interesting role, as will now be seen. The LB operator, Δ , on S^4 with this parameterization is

$$\Delta = \partial_\omega^2 + 3 \cot \omega \partial_\omega + (4/\sin^2 \omega)[\partial_\alpha^2 + \cot \alpha \partial_\alpha + (1/\sin^2 \alpha)(\partial_\beta^2 + \partial_\gamma^2 - 2 \cos \alpha \partial_\beta \partial_\gamma)].$$

The operator in brackets is the total angular momentum operator having the usual solutions, leaving

$$\Delta f(\omega) = f'' + 3 \cot \omega f' - [2\ell(2\ell + 2)/\sin^2 \omega]f = \delta(\omega) \quad (18)$$

where $f' = \partial f / \partial \omega$. The factor of four in the S^3 angular momentum operator is important in two ways: (i) by allowing half-integer spin while permitting a polynomial solution to the ω equation (as will be seen), and (ii) by embedding the S^3 solutions into S^4 , which is the role of the +2 in $2\ell(2\ell + 2)$.

The delta function on the right in eq. (18) represents a source at the pole, and this requires comment. The metric in eq. (17) is for the antipodal projection of S^4 onto \mathbb{R}^4 . In the Sp_2/Sp_1^2 picture, symmetry recommends that the two particles be placed at antipodal points of S^4 ; these two points map to $q = 0$ in the projection. Particles are singularities in classical electromagnetism, and high energy experimental evidence suggests that electrons are point-like particles. If one only looks for smooth solutions of eq. (18) there will be no distinguished points. The singularity at $\omega = 0$ is a statement of these general principles. Physical particles are points that are topologically equivalent to holes.

The solution of eq. (18) for $\ell = 0$ is $f_0 = -\cot \omega / \sin \omega + \ln[\tan(\omega/2)]$. This is a static and integrable potential that is continuous at $\omega = \pi/2$; continuity at the equator is essential to ensure continuity of the field on S^4 except at the poles. On attempting to solve the equation for $\ell \neq 0$ a singularity as $q \rightarrow \infty$ can be avoided by adding $\theta^2 f_\ell$ to the left in eq.

(18), which is equivalent to converting the operator to $(\Delta - \partial_t^2)f_\ell$, with $f_\ell = g_\ell \exp(i\theta t)$. One of the general solutions is

$$g_\ell = (\sin \omega)^{-2(\ell+1)} \sum_{n=0}^N a_n (\sin \omega)^{2n}$$

with

$$a_n = \frac{(2\ell - n)!(N - 2\ell - 3/2 + n)!}{n!(N - n)!}$$

and $\theta = \sqrt{(\ell + 1 - N)(\ell - 1/2 - N)}$. The polynomial terminates at $n = N$, where $N < \ell + 1$ if ℓ is an integer and $N < \ell - 1/2$ if ℓ is a half-integer. The solution is again singular at the origin, but the time dependence insures that it is continuous at $\omega = \pi/2$. The integer solutions should represent the potentials for bosons that mediate the interaction between the two fermions in a meson. The half-integer solutions are surrogates for "*-inos" in SUSY theory.

The ground state and excited states of a bare meson are encompassed in this solution. The meson is bare because interactions with the surroundings are encompassed by $Sp(n)/Sp(2) \times Sp(n-2)$; the solutions to eq. (18) presumably give a part of the energy, but interactions with the surroundings are required to get the total energy. (It will not have escaped the reader's attention that an operator giving the energy of a system has not yet been identified—it has not yet been found.) An interesting aspect of g_ℓ is that it is not integrable for $\ell > 1/2$, which may have some bearing on questions of stability. (With respect to SUSY, a matrix in $Sp(n)$ is constructed from eigenvectors and eigenvalues, which clearly provides the relation between interactions (bosons) and fermionic eigenvectors.)

INDUCED REPRESENTATIONS

It is probable that induced representation theory will be important in constructing representations with higher weights than the natural representation. The structure of induced representations is a perfect expression of the invariance of the inner product of wave functions under the gauge group. To set this up one simply notes that everything that has been done so far is aimed at describing interactions between subsystems regardless of their internal states. This suggests that one "average" over the gauge group. Let H be a maximal compact factor group (the gauge group) of $Sp(n)$. Put $\xi \in H$ and let $\sigma(\xi)$ be a unitary

representation of H . A function defined by

$$f_\theta(x) = \int_H \sigma(\eta) \theta(x\eta) d\eta$$

is a standard construction. Here $\theta(x)$ is a map from $Sp(n)$ to a Hilbert space with an inner product. It is easy to show that $f_\theta(x\xi) = \sigma(\xi^{-1})f_\theta(x)$ [9, 24]. That is, the inner product $\langle f, f \rangle$ is independent of H .

The vector bundles that represent physical systems in excited states have to be constructed from higher dimensional representations, but it seems that the theory wants the group to act in a form having the same dimension as the natural representation. A reasonable assumption is that $Sp(n)$ acts in this form and is to be constructed from matrix elements extracted from higher dimensional representations. The reason for inserting this condition is that a single fermion in an excited state must still be a quaternion; there was no restriction on the state of a particle in the construction of the flag manifold.

WHAT IS COLOR?

The fundamental operation of matrix multiplication: $a_{ij}a_{jk} \rightarrow a_{ik}$ (no sum), generates the third interaction from the product of two interactions in this triangular relation. This naive observation may be important in understanding the stability of three fermions (quarks). The flag $F_3 := Sp_3/Sp_1^3$ should correspond to $SU(3)$ with spin degrees of freedom explicit. The extra degrees of freedom in the flag manifold enable one to understand "apparent" violations of the exclusion principle. To see how this might be done, consider a particular representation of the Lie algebra of F_3 by

$$F_3 \sim \exp(g) = \exp \begin{bmatrix} 0 & \gamma \mathbf{k} & \bar{\beta} \mathbf{j} \\ \bar{\gamma} \mathbf{k} & 0 & \alpha \mathbf{i} \\ \beta \mathbf{j} & \bar{\alpha} \mathbf{i} & 0 \end{bmatrix}$$

where α, β, γ are \mathbb{C} -valued functions, with $\bar{\alpha}$, etc., being complex conjugates. This particular representation maps the three colors of chromodynamics to the quaternion basis. Using complex functions expands the quaternion algebra to biquaternions, which are almost, but not quite, isomorphic to the octonions. (The essential difference being that $\sqrt{-1}$ commutes with the quaternions whereas the fifth basis element of the octonions does not.) Note that g is indeed skew-symmetric because the quaternions are skew-symmetric. If the quaternions

are erased, the matrix appears to be an element in the algebra of $U(3)/[\pm\sqrt{-1} \times U(1)^3]$. In selecting a representation with each of the quaternion basis elements appearing just once, a notable structure emerges. The remarkable aspect of this representation is that

$$g^2 = \begin{bmatrix} -(|\beta|^2 + |\gamma|^2) & -\overline{\alpha\beta}\mathbf{k} & \alpha\gamma\mathbf{j} \\ \alpha\beta\mathbf{k} & -(|\alpha|^2 + |\gamma|^2) & -\overline{\beta\gamma}\mathbf{i} \\ -\overline{\alpha\gamma}\mathbf{j} & \beta\gamma\mathbf{i} & -(|\alpha|^2 + |\beta|^2) \end{bmatrix},$$

which suffices to show that the quaternion basis structure of this representation of the group is similar to that of its algebra. Furthermore, the form is maintained on permutation of the basis. This representation has some of the features of $SU(2) \times SU(3)$; and since the Sp_1^3 gauge group contains $U(1)$ subgroups, the symplectic group has all the machinery required to classify the baryons.

Let v diagonalize g by conjugation: $v^{-1}gv \rightarrow \lambda$. The same matrix diagonalizes F_3 by $v^{-1}F_3v = \exp(v^{-1}gv)$. The colorless eigenvalue equation is

$$(\alpha\beta\gamma + \overline{\alpha\beta\gamma}) [\lambda^3 + \rho^2\lambda + (\alpha\beta\gamma - \overline{\alpha\beta\gamma})] = 0,$$

where $\rho^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2$. It is tempting to restrict the intrinsic geometry to a rank two space (three points determine a plane), which is accomplished by setting $\alpha\beta\gamma - \overline{\alpha\beta\gamma} = 0$. With this restriction, the eigenvalues of F_3 are $\text{diag}[\mathbf{e}, \exp(i\rho)\mathbf{e}, \exp(-i\rho)\mathbf{e}]$. The vanishing of the imaginary part of $\alpha\beta\gamma$ gives a quantum condition: linear combinations of the three arguments of α, β and γ must be integer multiples of π . These considerations lead to yet another conjecture: The essential difference between leptons and quarks is that the former are quaternionic functions of four real functions, whereas the latter are quaternion functions of four complex functions. Interactions between real-real, real-complex, and complex-complex quaternions might map to the three interactions of particle theory. Further pursuit of these considerations is best left to others.

COSMOLOGICAL CONSEQUENCES

There are two broader aspects of the theory that merit discussion. The first of these is provided by one parameter representations of the group. These are geodesics[9], and the identification of the parameter with Galilean time has been introduced through the equation of motion, eq. (5). At $t = 0, \exp(t\mathbf{g}) = \mathbf{1}$; this is a unique point on the group manifold. At

this point the algebra \mathfrak{g} can exist, but the group matrix, the identity, is vacuous. However, at the next instant the off-diagonal elements of the group matrix become populated, looking very much like the big bang. Furthermore, it does not appear that anything interesting happens for $t < 0$, since changing the sign of t is equivalent to transposing \mathfrak{g} ; this is merely a matter of convention and has no physical content. The parameter t is Galilean time and it is also cosmological time.

Different parts of the universe are in highly excited states (obviously!) that are continuously exchanging excitation energy with one another. Given that the theory was formulated in terms of the natural action of the group on particles with spin, regardless of their state, this implies that particles in both excited states and ground states are described by quaternion line bundles. But there is nothing to prevent one from considering a collection of particles that are in the ground state, and there is no apparent restriction on the number of particles that can reside in the ground state of a system. The \mathbb{R}^4 Pauli exclusion principle is no longer operative, as was seen in the discussion of color. The group can accomodate apparent violations of the Pauli exclusion principle because our wave functions are functions of pairwise coordinates in a non-Euclidean space. The generalization of the exclusion principle is that no two entries in $\Psi = [\psi_1, \psi_2, \dots, \psi_n]$ may be identical, as one of the particles may be made to vanish by a linear combination that is contained in $Sp(n)$.

This leads to a second proposal that begs to be stated: a black hole is a massive ground state object. Particles that fall into the black hole must release excitation energy, and this would appear as Hawking radiation. As this happens the accreting black hole energizes everything around it by releasing energy. A ground state object is immediately apparent through its angular momentum that is conveyed so clearly in the Lie algebra. This angular momentum cannot vanish – to do so would be equivalent to the object not existing. Others will have to find its gravitational influence.

CONCLUSIONS

This has the appearance of being a comprehensive theory of interactions – its mathematical structure is extremely rich, spanning an enormous range of topics from Lie algebras to topology to Schubert varieties. During the presentation correspondences with geometries of current interest in particle theory have been discussed to illustrate how parts of

this structure have been seen in other settings. A well-defined fiber bundle structure has been established to treat many-body interactions, and the action of the symplectic group on the vector bundle provides new insights into the structure of matter. First and foremost amongst these is the notion that to infer the properties of the simplest particle beyond its intrinsic angular momentum it is necessary to couple the particle to its surroundings, not to the vacuum. A symplectic flag manifold supports these many-body quantum interactions.

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